Gentzen's unpublished normalization theorem and its successors (Proof theory and proving)

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1 Introduction

Gentzen mentioned in the synopsis of his paper [2] that he had first obtained his Hauptsatz by using natural deduction in the case of intuitionistic logic, but to settle the problem arising with the law of excluded middle in classical logic, he introduced another deduction system, i.e. sequent calculus, and proved the Hauptsatz as cut-elimination theorem on it. Therefore, his original proof of the Hauptsatz in the style of natural deduction had not been known. Concerning this deduction system, after Gentzen, Prawitz [6] proved the Hauptsatz as normalization theorem on intuitionistic natural deduction and also on the disjunction and existential quantifier free fragment of classical case, and later, in Stålmarck's work [7] and in some others' ones, including ours [1], Prawitz' proof was extended to full classical logic.

After these progression, some amount of Gentzen's handwritten notes were found and they were translated to English and published by von Plato [3, 4, 5]. The notes contains Gentzen's original proof of the Hauptsatz for intuitionistic logic on natural deduction. Comparing Gentzen's and Prawitz' proof for the same theorem, we can see that the two are essentially similar. In fact, Gentzen defines the notion *bond* which corresponds with Prawitz' *segment*, and for the *hillock* (maximum segment) of length more than one, Gentzen's reduction is identical with Prawitz' one, i.e. permutation conversion in the present expression. On the other hand, in the details. there is a little difference between their proofs, such as the way of choosing the hillock to be reduced first, and of divisions of successive reductions to decrease the induction value.

In this paper, we reform our previous proof [1] of normalization theorem for full classical natural deduction, as the extension of Gentzen's original proof for intuitionistic logic unveiled by von Plato [3, 4, 5]. We have some simplification of the assignment of induction value.

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2 Definitions

In our system of classical natural deduction; primitive logical symbols are &, $\vee$, $\forall$, $\exists$, $\supset$, and $\neg$; and inference rules are introduction rules $(*I)$ for each logical symbol $*$, elimination rules $(*E)$ for each logical symbol $*$, and the classical absurdity rule $(\perp_c)$ which is shown by the following schema.

$$\begin{array}{c}[\neg A]\\ \dfrac{\perp}{A}\,(\perp_c)\end{array}$$

We suppose all derivations are regular as in [1] without loss of generality, that is, each assumption discharged by a classical absurdity rule is the major premiss of an elimination rule (for negation).

The notion *main formula* and *neighbor* are defined as in Gentzen's notes (1.11 and 1.12 in [3]). We extend the notion *bonded* (1.13 in [3]) by adding the condition that the minor premiss of the $(\neg E)$ whose major premiss is discharged by a $(\perp_c)$, say I, and the conclusion of I are *bonded* with each other.

After the adaptation above of the notion "bonded" for our classical natural deduction, we use Gentzen's definitions in [3] without change, such as; *bond* (1.2), *representant* (1.21), *neighbor* as bonds (1.21), *hillock* (1.3), *grade* of a formula (1.3), and *higher* (1.5). The definition of *inner hillock* (1.3) is modified, for permitting open assumption formulas of derivations in our system, such that it is defined as a hillock to which neither the endformula nor an open assumption belongs. Then, we define similarly to (1.5) that "a *main hillock* is an inner hillock such that there is (in the derivation) no inner hillock of a greater grade, and no higher one of the same grade".

3 Normalization theorem

We have the normalization theorem, i.e. *hillock theorem* in Gentzen's terminology, for our classical natural deduction system.

Theorem. *For an arbitrary given derivation, we can transform it to a derivation of same assumptions and endformula with no inner hillock.*

Proof. By induction on $\langle g, n\rangle$, where g is the greatest grade of inner hillocks and n is the number of inner hillocks of the greatest grade in a derivation. Induction values are compared by lexicographical order.

Choose one of the main hillocks $\mathcal{H}$ in a given derivation Π with some inner hillocks. In the case that $\mathcal{H}$ consists of more than one formula, we apply for the lowest formula, say

C^0, of $\mathcal{H}$ the permutation conversion defined in 2.2 of [3], or in IV §I of [6], (if C^0 is the major premiss of a $(\vee E)$ or a $(\exists E)$) and in §3 of [1] (if C^0 is the major premiss of a $(\perp_c)$), the latter is described as:

$$
\dfrac{\dfrac{\begin{array}{c}\dfrac{[\neg C] \quad \begin{array}{c}\vdots\\ C\end{array}}{\perp}\\ \vdots\\ \perp\end{array}}{C^0}\,(\perp_c) \qquad \Sigma_1 \quad \Sigma_2}{D}
\qquad \longrightarrow \qquad
\dfrac{\begin{array}{c}\dfrac{[\neg D] \qquad \dfrac{\begin{array}{c}\vdots\\ C\end{array} \quad \Sigma_1 \quad \Sigma_2}{D}}{\perp}\\ \vdots\\ \perp\end{array}}{D}\,(\perp_c) ,
$$

successively until $\mathcal{H}$ has been shrunken to one single formula, i.e. to its representant. In the course of this application of permutation conversions, the induction value of the derivation does not increase because there is no representant of inner hillock of grade g above or equal to the minor premiss of the inference whose major premiss is C^0. With this preparatory steps concerning permutation conversions, we can now suppose $\mathcal{H}$ consists one single formula. By applying the proper reduction of the main hillock defined in 2.4 of [3] as usual, we have a derivation whose induction value is less than that of Π because of the conditions for the representant of the main hillock. □

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